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"A priori" estimates, uniqueness and existence of positive solutions of Yamabe type equations on complete manifolds

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Abstract

We study the asymptotic behaviour of non-negative solutions of Yamabe type equations on a complete Riemannian manifold. Then we provide a comparison result, based on a form of the weak maximum principle at infinity, which together with the "a priori" estimates previously obtained, yields uniqueness under very general Ricci assumptions. The paper ends with an existence result and an application to the non-compact Yamabe problem.

Keywords: "A priori" estimates; Uniqueness; Existence; Yamabe type equations; Riemannian manifolds

1. Introduction

The aim of this paper is to study the asymptotic behaviour, uniqueness and existence of positive solutions of the equation

$$\Delta u + a(x)u - b(x)u^{\sigma} = 0 \tag{1.1}$$

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on a complete, non-compact, connected Riemannian manifold $(M, \langle\,,\rangle)$. Here $\sigma > 1$ and the coefficient $b(x)$ is assumed to be non-negative, while $a(x)$ is not assumed to be of constant sign.

Equations of the form (1.1) arise, for instance, in Riemannian geometry as the equation for the change of the scalar curvature under a conformal change of metric and in mathematical biology, where they describe the steady state solutions of the logistic equation with diffusion

$$\frac{\partial u}{\partial t} = \Delta u + a(x)u - b(x)u^{\sigma}.$$

In this latter contest u represents the density of a population and it is therefore assumed to be non-negative. For a detailed discussion of these two examples we refer, for instance, respectively to [7,13] and the recent [4,5,12].

From now on we fix an origin $o \in M$ and let $r(x) = dist(x, o)$. We set $B_R = \{x \in M\colon r(x) < R\}$ for the geodesic ball of radius R centred at o.

Estimates from above, for positive solutions of (1.1), under assumptions on the Ricci curvature of M of the type

$$Ricci_{(M,\langle\,,\rangle)} \geqslant -(m-1)H^2\bigl(1+r^2(x)\bigr)^{\frac{\delta}{2}} \tag{1.2}$$

in the sense of quadratic forms (the metric is understood in the right-hand side of (1.2)), for some $H, \delta \in \mathbb{R}$, and appropriate bounds on the behaviours of $a(x)$, $b(x)$ at infinity, have been given sometime ago in [13] and refined in a subsequent series of papers. On the other hand, despite of the number of results on Euclidean space $\mathbb{R}^m$, see, for instance, [3,4] and the references therein, we are not aware of estimates on $u > 0$ from below in the above generality. Indeed, it goes without saying that the Euclidean techniques, for instance rescaling together with the well-known squeezing method of [4], are fruitless in yielding interesting conclusions under geometric assumptions of the type (1.2) on a general manifold.

Using the approach developed in [13], we provide a sharp result in this setting in Theorem 2.1. As an immediate application, Proposition 2.2 yields completeness of the conformally deformed metric in the non-compact Yamabe problem. See the seminal papers [8,9] and again [7] for a thorough discussion.

Uniqueness is obtained in Theorem 3.3 via a comparison result, Theorem 3.1, of independent interest. The proof of this latter is based on a form of the weak maximum principle at infinity as introduced in [11] and [14]. In this way we are able to replace curvature assumption (1.2) with the weaker volume growth type condition

$$\liminf_{R\to+\infty} \frac{\log vol\, B_R}{R^{\xi}} < +\infty \tag{1.3}$$

for some appropriate $\xi > 0$ related to the behaviour at infinity of $a(x)$, $b(x)$ and to that of the two solutions u and v of (1.1) to be compared. A counterexample after the proof shows that the requests on u and v cannot be relaxed. As before previous techniques and results in Euclidean space, see for instance [4], are not applicable in this general case. Putting together the “a priori” estimates of Theorems 2.1 and 2.3 with uniqueness of Corollary 3.2 and estimates on the volume growth proved in [1] we finally obtain our main uniqueness result Theorem 3.3.

The paper ends with an existence result, Theorem 4.2, for Eq. (1.1) that we also interpret in the setting of the non-compact Yamabe problem. This compares with previous work of one of us. See [13, Theorem 1.2]. Further comments which relate the result to the literature follow the statement of the theorem.

2. “A priori” estimates

In this section we determine “a priori” estimates on the behaviour at infinity of positive solutions of the equation

$$\Delta u + a(x)u - b(x)u^{\sigma} = 0, \quad \sigma > 1, \tag{2.1}$$

on M under assumptions on $a(x)$ and $b(x)$ related to the geometrical requirement

$$Ricci_{(M,\langle\,,\,\rangle)} \geqslant -(m-1)H^2\big(1 + r(x)^2\big)^{\frac{\delta}{2}} \quad \text{on } M \tag{2.2}$$

for some $H > 0$ and $\delta \in \mathbb{R}$. We begin with

Theorem 2.1. *Let $(M, \langle\,,\,\rangle)$ be a complete manifold with Ricci tensor satisfying* (2.2) *and let $a(x)$ and $b(x) \in C^0(M)$ with $b(x) > 0$ on M and*

$$\liminf_{r(x)\to+\infty} \frac{a(x)}{r(x)^{\alpha}} > 0 \tag{2.3}$$

with $\alpha > \max\{-2, \delta\}$. Let $\psi(t)$ be a positive, non-decreasing function defined in a neighbourhood of infinity such that, for some $\epsilon \in (0, 1)$,

$$\psi(t) = O\left(\psi\left(\frac{t}{1+\epsilon}\right)\right) \quad \text{as } t \to +\infty \tag{2.4}$$

and assume that

$$\liminf_{r(x)\to+\infty} \frac{a(x)}{b(x)}\psi\big(r(x)\big) > 0. \tag{2.5}$$

Then, any positive solution $u \in C^2(M)$ of

$$\Delta u + a(x)u - b(x)u^{\sigma} \leqslant 0, \quad \sigma > 1, \quad \text{on } M \tag{2.6}$$

satisfies

$$u(x) \geqslant C\psi\big(r(x)\big)^{-\frac{1}{\sigma-1}} \tag{2.7}$$

for $r(x) \gg 1$ and some constant $C > 0$.

Proof. We fix $q \in M$, with $r(q) \gg 1$ and we set $\rho(x) = dist(x, q)$. Fix $T > 0$ and consider on $B_T(q)$ the function

$$F(x) = \frac{u(x)}{[T^2 - \rho^2(x)]^{\xi}}$$

for some $\xi > 1$. Note that, as $x \to \partial B_T(q)$, $F(x) \to +\infty$, while $F(x) > 0$ on $B_T(q)$. Thus F attains a positive absolute minimum at $\bar{x} \in B_T(q)$. Using a trick of Calabi [2], which enables us to suppose that ρ is smooth near $\bar{x}$, we deduce

$$\text{(i)}\quad \nabla \log F(\bar{x}) = 0, \qquad \text{(ii)}\quad \Delta \log F(\bar{x}) \geqslant 0. \tag{2.8}$$

Using (2.8)(i) a computation yields

$$\frac{\nabla u}{u}(\bar{x}) = -2\xi \frac{\rho \nabla \rho}{T^2 - \rho^2}(\bar{x}), \tag{2.9}$$

while from (2.8)(ii) and (2.9) we have at $\bar{x}$

$$\frac{\Delta u}{u} - 4\xi(\xi - 1)\frac{\rho^2}{[T^2 - \rho^2]^2} + \xi \frac{\Delta \rho^2}{T^2 - \rho^2} \geqslant 0. \tag{2.10}$$

In order to estimate $\Delta \rho^2$, let

$$Ricci_{(M, \langle\, ,\, \rangle)} \geqslant -(m-1)Z^2 \quad \text{on } \overline{B_T(q)}. \tag{2.11}$$

Therefore, from the Laplacian comparison theorem we infer

$$\Delta \rho^2 \leqslant 2\big[m + (m-1)Z\rho\big] \quad \text{on } \overline{B_T(q)}.$$

Inserting (2.6) and (2.11) into (2.10) we have at $\bar{x}$

$$u \geqslant b^{-\frac{1}{\sigma-1}} \left\{ a - 4\xi(1-\xi)\frac{\rho^2}{[T^2 - \rho^2]^2} - 2\xi \frac{m + (m-1)Z\rho}{T^2 - \rho^2} \right\}^{\frac{1}{\sigma-1}}.$$

Since $\bar{x}$ is the minimum of F on $B_T(q)$ we then deduce

$$\begin{aligned}
\big[T^2 - \rho^2(\bar{x})\big]^{\xi} u(y) &\geqslant \big[T^2 - \rho^2(y)\big]^{\xi} u(\bar{x}) \\
&\geqslant \big[T^2 - \rho^2(y)\big]^{\xi} b(\bar{x})^{-\frac{1}{\sigma-1}} \left\{ a(\bar{x}) + 4\xi(\xi - 1)\frac{\rho^2(\bar{x})}{[T^2 - \rho^2(\bar{x})]^2} \right. \\
&\qquad \left. - 2\xi \frac{m + (m-1)Z\rho(\bar{x})}{T^2 - \rho^2(\bar{x})} \right\}^{\frac{1}{\sigma-1}}
\end{aligned}$$

for each $y \in B_T(q)$. In particular, for $y = q$,

$$u(q) \geqslant \left[\frac{a(\bar{x})}{b(\bar{x})}\right]^{\frac{1}{\sigma-1}} \left\{ 1 + \frac{4\xi(\xi-1)}{a(\bar{x})} \frac{\rho^2(\bar{x})}{[T^2 - \rho^2(\bar{x})]^2} - \frac{2\xi}{a(\bar{x})} \frac{m + (m-1)Z\rho(\bar{x})}{T^2 - \rho^2(\bar{x})} \right\}^{\frac{1}{\sigma-1}}. \tag{2.12}$$

We set

$$f(t) = \frac{1}{2} + \frac{2\xi(\xi-1)}{a(\bar{x})} \frac{t^2}{[T^2 - t^2]^2} - \frac{2\xi}{a(\bar{x})} \frac{(m-1)Zt}{T^2 - t^2}$$

and

$$g(t)=\frac{1}{2}+\frac{2\xi(\xi-1)}{a(\bar{x})}\frac{t^2}{[T^2-t^2]^2}-\frac{2\xi}{a(\bar{x})}\frac{m}{T^2-t^2}$$

on $[0,T)$ and, considering the parabola (note $\xi>1$)

$$w=\frac{2\xi(\xi-1)}{a(\bar{x})}y^2-\frac{2\xi(m-1)Z}{a(\bar{x})}y+\frac{1}{2},\quad y\in[0,+\infty),$$

we deduce that $f(t)$ attains on $[0,T)$ its minimum value

$$\bar{f}=\frac{1}{2}-\frac{1}{2}\frac{\xi}{\xi-1}\frac{(m-1)^2Z^2}{a(\bar{x})}.$$

As for $g(t)$, we have $g(0)=\frac{1}{2}-\frac{2\xi}{a(\bar{x})}\frac{m}{T^2}$, $\lim_{t\to T^-}g(t)=+\infty$ and

$$\begin{aligned}g'(t)&=\frac{4\xi}{a(\bar{x})}\frac{t}{[T^2-t^2]^3}\left\{(\xi-1-m)t^2+(\xi-1+m)T^2\right\}\\&\geqslant\frac{8}{a(\bar{x})}\frac{t}{[T^2-t^2]^3}\xi(\xi-1)\\&\geqslant 0\end{aligned}$$

on $[0,T)$ because of our choice of ξ. It follows that $g(t)$ attains on $[0,T)$ its minimum value

$$\bar{g}=\frac{1}{2}-\frac{2\xi}{a(\bar{x})}\frac{m}{T^2}.$$

Going back to (2.12) we obtain

$$u(q)\geqslant\left[\frac{a(\bar{x})}{b(\bar{x})}\right]^{\frac{1}{\sigma-1}}\left\{1-\frac{\xi}{2(\xi-1)}\frac{(m-1)^2Z^2}{a(\bar{x})}-\frac{2\xi}{a(\bar{x})}\frac{m}{T^2}\right\}^{\frac{1}{\sigma-1}}.\tag{2.13}$$

We now choose

$$T=\epsilon r(q)$$

and we observe that, since $\forall x\in B_T(q)$, $r(q)-T\leqslant r(x)\leqslant r(q)+T$, with our choice of T, using (2.3) we have

$$\begin{aligned}&a(x)\geqslant C(\epsilon)r(q)^{\alpha},\\&Ricci_{(M,\langle\,,\rangle)}\geqslant-(m-1)H^2\left[1+W(\epsilon)^2r(q)^2\right]^{\frac{\delta}{2}}\end{aligned}\tag{2.14}$$

for some constants $C(\epsilon)$, $W(\epsilon)>0$, depending only on $\epsilon>0$ and on the sign of α and δ, respectively. Therefore,

$$\begin{aligned}\Lambda(\bar{x})^{\sigma-1} &= 1 - \frac{\xi}{2(\xi-1)}\frac{(m-1)^2 Z^2}{a(\bar{x})} - \frac{2\xi}{a(\bar{x})}\frac{m}{T^2} \\ &\geqslant 1 - \frac{\xi(m-1)^2 H^2}{2(\xi-1)C(\epsilon)}\left[\frac{1}{r^2(q)} + W(\epsilon)^2\right]^{\frac{\delta}{2}} r(q)^{\delta-\alpha} - \frac{2m\xi}{C(\epsilon)}\frac{1}{\epsilon^2} r(q)^{-2-\alpha}. \qquad (2.15)\end{aligned}$$

Thus, using the assumptions $\alpha > -2$, $\alpha > \delta$ we can suppose to have chosen $R > 0$ sufficiently large such that, $\forall q$ with $r(q) > R$, we have $\Lambda(\bar{x}) \geqslant \frac{1}{2}$. Hence

$$u(q) \geqslant \frac{1}{2}\left[\frac{a(\bar{x})}{b(\bar{x})}\right]^{\frac{1}{\sigma-1}}$$

and in turn

$$u(q)\psi\big(r(q)\big)^{\frac{1}{\sigma-1}} \geqslant \frac{1}{2}\left[\psi\big(r(q)\big) \min_{B_T(q)} \frac{a(x)}{b(x)}\right]^{\frac{1}{\sigma-1}}. \qquad (2.16)$$

We claim that

$$\liminf_{r(q)\to+\infty} \psi\big(r(q)\big) \min_{B_T(q)} \frac{a(x)}{b(x)} > 0 \qquad (2.17)$$

so that (2.7) follows at once from (2.16). To prove the claim suppose that (2.17) is false, then there exists a sequence $\{y_n\}$ in M, with $r(y_n) \to +\infty$, such that

$$\lim_{r(y_n)\to+\infty} \psi\big(r(y_n)\big) \min_{B_{T_n}(y_n)} \frac{a(x)}{b(x)} = 0, \qquad (2.18)$$

where $T_n = \epsilon r(y_n)$. Let $\{z_n\} \in \overline{B_{T_n}(y_n)}$ realize the minimum, that is

$$\min_{B_{T_n}(y_n)} \frac{a(x)}{b(x)} = \frac{a(z_n)}{b(z_n)}.$$

Fix $\eta > 0$ and choose n sufficiently large so that, from (2.18), we have

$$\frac{a(z_n)}{b(z_n)} < \frac{\eta}{\psi(r(y_n))}. \qquad (2.19)$$

Since $z_n \in B_{T_n}(r(y_n))$,

$$r(y_n) \geqslant \frac{r(z_n)}{1+\epsilon}$$

and therefore, since ψ is non-decreasing and (2.4) holds we have

$$\frac{a(z_n)}{b(z_n)} < \frac{\eta}{\psi(\frac{r(z_n)}{1+\epsilon})} \leqslant \frac{C\eta}{\psi(r(z_n))} \qquad (2.20)$$

for some constant $C > 0$ independent of n. In other words

$$\psi\big(r(z_n)\big)\frac{a(z_n)}{b(z_n)} \leqslant C\eta$$

for $n \gg 1$. Since $\eta > 0$ was chosen arbitrarily this contradicts (2.5) and completes the proof of the theorem. $\square$

Remark 2.1. 1. If we assume $\psi(t)$ non-increasing the conclusion of Theorem 2.1 holds substituting the request (2.4) with

$$\psi(t) = O\left(\psi\left(\frac{t}{1-\epsilon}\right)\right) \quad \text{as } t \to +\infty. \tag{2.21}$$

Thus Theorem 2.1 is valid in particular with $\psi(t) = Ct^{\beta}(\log t)^{\gamma}$ for $t \gg 1$, $\beta, \gamma \in \mathbb{R}$, $C > 0$ and $\epsilon \in (0, 1)$.

With this observation and $\delta = -2$ in (2.2), we recover the estimate from below on $\mathbb{R}^m$ of Dong [3, Theorem 1.1].

2. Clearly, (2.4) (or (2.21)) are not satisfied if $\psi(t)$ is of exponential type, for instance $\psi(t) = Ce^{\beta t}$, $C, \beta > 0$. In this case condition (2.4) has to be substituted with

$$\psi(t) = O\big(\psi(t - T_0)\big) \quad \text{as } t \to +\infty \tag{2.22}$$

for some $T_0 > 0$. The result continues to hold with the request

$$\alpha > \max\{0, \delta\} \tag{2.23}$$

but the proof has to be modified as follows. Following the argument of Theorem 2.1 we arrive at (2.13). Now we choose $T = T_0$, then (2.14) holds with $C(\epsilon)$, $W(\epsilon)$ substituted with $C(T_0)$, $W(T_0)$ positive constants depending only on T_0 and the sign of α and δ, respectively. We then proceed, under the modified assumption (2.3) with $\alpha > \max\{0, \delta\}$, directly to

$$u(q)\psi\big(r(q)\big)^{\frac{1}{\sigma-1}} \geqslant \frac{1}{2}\left[\psi\big(r(q)\big)\min_{B_{T_0}(q)}\frac{a(x)}{b(x)}\right]^{\frac{1}{\sigma-1}}. \tag{2.24}$$

The remaining of the proof is the same as in Theorem 2.1 using (2.22) instead of (2.4).

Clearly if $\psi(t)$ is non-increasing then (2.22) has to be substituted with

$$\psi(t) = O\big(\psi(t + T_0)\big) \quad \text{as } t \to +\infty \tag{2.25}$$

for some $T_0 > 0$.

Remark 2.2. In case $\alpha = \delta$ the proof of Theorem 2.1 can be modified to obtain the same conclusion provided a further condition on H is satisfied.

Indeed, in case (2.4) or (2.21) holds, assume that the Ricci tensor of M satisfies (2.2) with $\delta > -2$, and having set

$$A = \liminf_{r(x)\to+\infty}\frac{a(x)}{r(x)^{\alpha}} > 0,$$

suppose that H satisfies

$$0 < H^2 < \frac{A}{(m-1)^2}\left(\frac{1-\epsilon}{1+\epsilon}\right)^{|\delta|}. \tag{2.26}$$

Under these assumptions, following the proof of Theorem 2.1, we arrive at (2.15) which takes the form

$$\Lambda(\bar{x})^{\sigma-1} \geqslant 1 - \frac{\xi}{2(\xi-1)}\frac{(m-1)^2H^2}{C(\epsilon)}\left[\frac{1}{r^2(q)} + W(\epsilon)^2\right]^{\frac{\delta}{2}} - \frac{2m\xi}{C(\epsilon)}\frac{1}{\epsilon^2}r(q)^{-2-\delta}$$

where $C(\epsilon)$ and $W(\epsilon)$ are given by

$$\begin{cases} C(\epsilon) = A\big(1-(\operatorname{sign}\delta)\epsilon\big)^{\delta}, \\ W(\epsilon) = \big(1+(\operatorname{sign}\delta)\epsilon\big). \end{cases} \tag{2.27}$$

Using (2.26) and $\delta > -2$ we can choose $\xi > 1$ and $R > 0$ both sufficiently large that, for some $\eta \in (0,1)$, $r(q) > R$ gives

$$\frac{\xi}{(\xi-1)}\frac{(m-1)^2H^2}{C(\epsilon)}\left[\frac{1}{r^2(q)} + W(\epsilon)^2\right]^{\frac{\delta}{2}} < \eta$$

and

$$1 - \frac{2m\xi}{C(\epsilon)}\frac{1}{\epsilon^2}r(q)^{-2-\delta} \geqslant \eta.$$

Thus,

$$\Lambda(\bar{x})^{\sigma-1} \geqslant \frac{\eta}{2}.$$

The remaining of the proof is as above.

If $\psi(t)$ satisfies (2.22) or (2.25), since $\delta > 0$ because of (2.23) and $\alpha = \delta$, we set

$$0 < H^2 < \frac{A}{(m-1)^2}\frac{(1-T_0/R_0)^{\delta}}{(1+T_0/R_0)^{\delta}}$$

for $R_0 > T_0$ and the theorem is still valid.

We apply this remark to Yamabe equation to show, for instance, that the conformally deformed metric $u^{\frac{4}{m-2}}\langle\,,\rangle$, $m \geqslant 3$, is still complete provided $\langle\,,\rangle$ is so. A simple reasoning shows (see [13]) that this is the case if $u(x)$ is bounded below by a radial function $\tilde{\psi}(r) \notin L^{\frac{2}{m-2}}(+\infty)$. Thus one can choose $\psi(r)$ in (2.7) given by $\psi(r) = r^2(\log r)^2$ and deduce the following.

Proposition 2.2. *Let $(M, \langle\, ,\rangle)$ be a complete manifold of dimension $m \geqslant 3$ and Ricci tensor satisfying* (2.2) *with $\delta > -2$. Suppose the scalar curvature $s(x)$ verifies*

$$s(x) \leqslant -Ar^{\delta}(x)$$

for $r(x) \gg 1$ and some $A > 0$, and that

$$0 < H^2(m-1)^2 < A. \tag{2.28}$$

Let $K(x) \in C^{\infty}(M)$, $K(x) < 0$ on M and

$$K(x) \geqslant -Cr(x)^{\delta+2}\log^2 r(x)$$

for $r(x) \gg 1$ and some $C > 0$. Then any conformal deformation of $\langle\, ,\rangle$ to a new metric of scalar curvature $K(x)$ is complete.

Proof. If $u^{\frac{4}{m-2}}\langle\, ,\rangle$ is a conformal deformation of $\langle\, ,\rangle$ with scalar curvature $K(x)$ then $u > 0$ satisfies Yamabe equation

$$\frac{4(m-1)}{m-2}\Delta u - s(x)u + K(x)u^{\frac{m+2}{m-2}} = 0.$$

Now apply the results described above observing that for some $\epsilon > 0$ sufficiently small (2.26) is satisfied and that for the same ϵ, (2.4) holds because of our choice of $\psi(r)$. $\square$

Note that the proof of Theorem 2.1 gives no uniform positive lower bounds on compact domains for the positive solutions of (1.1). This contrasts with the case of the estimate from above obtained in [13], see also [11], that we recall in a simplified form that will be most useful in the sequel.

Theorem 2.3. *Let $(M, \langle\, ,\rangle)$ be a complete manifold with Ricci tensor satisfying* (2.2) *and $\delta \geqslant -2$. Let $a(x), b(x) \in C^0(M)$ and satisfying*

$$a(x) \leqslant Ar(x)^{\alpha}, \quad \alpha \geqslant \frac{\delta}{2} - 1, \tag{2.29}$$

$$b(x) \geqslant Br(x)^{\beta}, \quad \beta \leqslant 1 - \frac{\delta}{2} + \alpha \tag{2.30}$$

for $r(x) \gg 1$ and some constants $A, B > 0$. Then any non-negative solution $u \in C^2(M)$ of

$$\Delta u + a(x)u - b(x)u^{\sigma} \geqslant 0, \quad \sigma > 1, \quad \text{on } M, \tag{2.31}$$

satisfies

$$u(x) \leqslant Cr(x)^{-\frac{\beta-\alpha}{\sigma-1}} \tag{2.32}$$

for $r(x) \gg 1$ and some constant $C > 0$.

As for the sharpness of the estimates obtained in Theorems 2.1 and 2.3 for this latter we refer, for instance, to [11, Remark 2.11, p. 35], while for Theorem 2.1 we provide the following example.

Let $g(r) \in C^\infty([0, +\infty])$, $g(r) > 0$ for $r > 0$ be such that

$$g(r) = \begin{cases} r & \text{on } [0, \frac{1}{2}], \\ e^{\frac{1}{m-1}\int_0^r (1+s^2)^{\delta/4}\, ds} & \text{on } [1, +\infty) \end{cases}$$

for some $\delta \geqslant -2$. We define the model manifold $M_g = \mathbb{R}^m$ in the sense of Greene and Wu [6], with metric on $M_g \setminus \{0\} = (0, +\infty) \times S^{m-1}$

$$\langle\, ,\rangle = dr^2 + g(r)^2\, d\theta^2,$$

where $d\theta^2$ is the canonical metric on the unit sphere S^{m-1}. Note that, since $g(r) = r$ on $[0, \frac{1}{2}]$, $\langle\, ,\rangle$ can be smoothly extended to all of M_g. The Ricci curvature in the radial direction is given by

$$Ricci_{(M,\langle\, ,\rangle)}(\nabla r, \nabla r) = -(m-1)\frac{g''(r)}{g(r)}$$

while, in the direction orthogonal to ∇r determined by the unit vector ν,

$$Ricci_{(M,\langle\, ,\rangle)}(\nu, \nu) = -\frac{(m-2)}{g(r)^2}\bigl(1 - g'(r)^2\bigr) - \frac{g''(r)}{g(r)}.$$

Thus, a simple computation shows that there exists an appropriate $H > 0$ such that

$$Ricci_{(M,\langle\, ,\rangle)} \geqslant -(m-1)H^2\bigl(1 + r^2\bigr)^{\frac{\delta}{2}} \quad \text{on } M_g.$$

Next, we consider the function

$$w(r) = \bigl(\mu + P(r)\bigr)^{\frac{\alpha-\beta}{\sigma-1}} > 0 \quad \text{on } [0, +\infty),$$

where $\mu > 0$, $\beta \geqslant \alpha > \delta$ and $P \in C^2([0, +\infty))$ satisfies

$$P(r) = r, \quad \text{for } r \gg 1, \qquad P(r) \geqslant 0 \quad \text{on } [0, +\infty), \qquad P'(0) = 0.$$

We define

$$H_w(r) = w^{-\sigma}\left\{w'' + (m-1)\frac{g'(r)}{g(r)}w' + A(1+r)^\alpha w\right\}$$

for some $A > 0$ constant. Computing we have,

$$(1+r)^{-\beta}H_w(r) = \frac{(\mu+P(r))^{\beta-\alpha}}{(1+r)^{\beta-\alpha}}\left\{\frac{\alpha-\beta}{\sigma-1}\left[\frac{P''(r)}{(\mu+P(r))(1+r)^{\alpha}} + \frac{\alpha-\beta-\sigma+1}{\sigma-1}\frac{P'(r)}{(\mu+P(r))^2(1+r)^{\alpha}} + (m-1)\frac{g'(r)}{g(r)}\frac{P'(r)}{(\mu+P(r))(1+r)^{\alpha}}\right] + A\right\}.$$

Next, we note that

$$(m-1)\frac{g'(r)}{g(r)} = \left(1+r^2\right)^{\frac{\delta}{4}}$$

on $[1,+\infty)$, and therefore

$$(m-1)\frac{g'(r)}{g(r)}\frac{P'(r)}{(\mu+P(r))(1+r)^{\alpha}} = \frac{P'(r)}{(\mu+P(r))}\left(1+r^2\right)^{\frac{\delta}{4}}(1+r)^{-\alpha} \tag{2.33}$$

on $[1,+\infty)$.

Furthermore, the left-hand side of (2.33) is bounded near zero since $P'(0)=0$. It follows that, up to choosing $\mu \gg 1$, since $P(r)=r$ for $r(x) \gg 1$, we can choose $B>0$ such that

$$H_w(r) \leqslant B(1+r)^{\beta}$$

on $[0,+\infty)$. Recalling the definition of $H_w(r)$, we then have

$$w'' + (m-1)\frac{g'(r)}{g(r)}w' + A(1+r)^{\alpha}w - B(1+r)^{\beta}w^{\sigma} \leqslant 0. \tag{2.34}$$

Setting $u(x)=w(r(x))$, $a(x)=A(1+r(x))^{\alpha}$ and $b(x)=B(1+r(x))^{\beta}>0$ on M_g we then deduce

$$\Delta u + a(x)u - b(x)u^{\sigma} \leqslant 0.$$

Moreover,

$$\frac{a(x)}{b(x)} = \frac{A}{B}\left(1+r(x)\right)^{\alpha-\beta}$$

on M_g and we can therefore choose $\psi(t)=t^{\beta-\alpha}$. Since $\beta \geqslant \alpha$, ψ is non-decreasing, satisfies (2.4), (2.5) and according to Theorem 2.1,

$$u(x) \geqslant Cr(x)^{\frac{\alpha-\beta}{\sigma-1}} \tag{2.35}$$

for $r(x) \gg 1$ and some constant $C>0$. Since $u(x)=(\mu+r(x))^{\frac{\alpha-\beta}{\sigma-1}}$ for $r(x) \gg 1$, estimate (2.35) cannot be improved.

We observe that the estimate (2.7) of Theorem 2.1 on $u(x) > 0$ solution of (2.1) on M is heavily based on assumption (2.3) which we rewrite in the form

$$a(x) \geqslant Ar(x)^\alpha \quad \text{on } M \setminus B_{R_0}$$

for some $A, R_0 > 0$. If we relax the above to

$$a(x) \geqslant -Ar(x)^\alpha \quad \text{on } M \setminus B_{R_0} \tag{2.36}$$

the proof of Theorem 2.1 fails. However, it turns out that in this case a, at first sight, similar estimate, depending however directly on Ricci (in the radial direction), can be obtained in an elementary way with the aid of the maximum principle. Indeed, we have

Theorem 2.4. *Let $(M, \langle\,,\rangle)$ be a complete manifold with Ricci tensor satisfying*

$$Ricci_{(M,\langle\,,\rangle)}\langle\nabla r, \nabla r\rangle \geqslant -(m-1)H^2\big(1+r(x)^2\big)^{\frac{\delta}{2}} \tag{2.37}$$

on M for some $H > 0$, $\delta > -2$ and let $a(x)$ and $b(x) \in C^0(M)$ satisfy, for $r(x) \gg 1$, (2.36) and

$$b(x) \leqslant Br(x)^\beta e^{(\sigma-1)\gamma(1+r(x)^{\frac{\delta+2}{2}})} \tag{2.38}$$

for some $A, B > 0$, $\alpha < \delta$, $\beta \leqslant \delta$, $\gamma > \frac{2H}{2+\delta}$, $\sigma > 1$. Then any positive solution of

$$\Delta u + a(x)u - b(x)u^\sigma \leqslant 0 \quad \text{on } M \tag{2.39}$$

satisfies

$$u(x) \geqslant Ce^{-\gamma r(x)^{\frac{\delta+2}{2}}} \tag{2.40}$$

for $r(x) \gg 1$ and some appropriate constant $C > 0$.

Proof. First of all, using Proposition 5.1 in [1], the Laplacian comparison theorem and (2.37) we deduce

$$\Delta r \leqslant Hr^{\frac{\delta}{2}}\big(1+o(1)\big) \quad \text{as } r \to +\infty. \tag{2.41}$$

Next we choose $R > 0$ sufficiently large so that (2.36) and (2.38) hold outside B_R and let $0 < \xi < \min_{\overline{B_R}} u(x)$. To simplify the writing, set $\theta = \frac{\delta+2}{2}$ and define

$$v(x) = \xi e^{-\gamma(1+r(x)^\theta)} - u(x).$$

Note that, since $\gamma > 0$, by our choice of ξ, we have

$$v(x) \leqslant \xi - u(x) < 0 \tag{2.42}$$

on $\overline{B_R}$. We now reason by contradiction and we suppose that

$$\liminf_{r(x)\to+\infty} \frac{u(x)}{e^{-\gamma(1+r(x)^\theta)}} = 0.$$

This means that there exists a sequence $\{x_n\} \subset M$, $r(x_n) \to +\infty$, such that

$$\frac{u(x_n)}{e^{-\gamma(1+r(x_n)^\theta)}} \to 0, \quad \text{as } n \to +\infty. \tag{2.43}$$

Since $\gamma, \xi > 0$ and $u(x) > 0$, $v^* = \sup_M v(x) < +\infty$. Furthermore, because of (2.43)

$$v(x_n) = e^{-\gamma(1+r(x_n)^\theta)} \left\{ \xi - \frac{u(x_n)}{e^{-\gamma(1+r(x_n)^\theta)}} \right\} > 0 \tag{2.44}$$

for n sufficiently large. Thus $v^* > 0$. Finally $\gamma, \theta > 0$ force that v^* has to be attained at some point $\bar{x} \in M$. Let Ω be the level set

$$\Omega = \bigl\{x \in M\colon\ v(x) > 0\bigr\}.$$

Because of (2.42), $\Omega \subset M \setminus \overline{B_R}$ and

$$u(x) < \xi e^{-\gamma(1+r(x)^\theta)} \quad \text{on } \Omega. \tag{2.45}$$

We compute on Ω

$$\Delta v = \xi\bigl[-\gamma\theta e^{-\gamma(1+r(x)^\theta)} r^{\theta-1}\Delta r + \gamma\theta\bigl(\gamma\theta r^{2\theta-2} + (1-\theta)r^{\theta-2}\bigr)e^{-\gamma(1+r(x)^\theta)}\bigr] - \Delta u$$

so that, using (2.36), (2.37), (2.41), (2.45) and by (2.39),

$$\begin{aligned}
\Delta v &\geqslant \xi r^{\frac{\delta}{2}-1+\theta} e^{-\gamma(1+r(x)^\theta)} \bigl\{\gamma\theta\bigl[\gamma\theta r^{\theta-1-\frac{\delta}{2}} + (1-\theta)r^{-1-\frac{\delta}{2}} - H\bigl(1+o(1)\bigr)\bigr] \\
&\quad - A r^{\alpha-\theta+1-\frac{\delta}{2}} - \xi^{\sigma-1} B r^{\beta-\theta+1-\frac{\delta}{2}}\bigr\} \\
&= \xi r^\delta e^{-\gamma(1+r(x)^{\frac{\delta+2}{2}})} \left\{ \frac{\delta+2}{2}\gamma\left[\frac{\delta+2}{2}\gamma - \frac{\delta}{2} r^{-1-\frac{\delta}{2}} - H\bigl(1+o(1)\bigr)\right] \right. \\
&\quad \left. - A r^{\alpha-\delta} - \xi^{\sigma-1} B r^{\beta-\delta} \right\}.
\end{aligned}$$

Note that, because of our assumptions on $\alpha, \beta, \gamma, \delta$ we can choose R sufficiently large and $\xi > 0$ sufficiently small that

$$\Delta v > 0 \quad \text{on } \Omega,$$

contradicting the fact that $\bar{x} \in \Omega$. $\square$

The case $\delta = -2$ is “loosely speaking” border line between Euclidean and hyperbolic geometry. For instance the assumption

$$Ricci_{(M,\langle\,,\,\rangle)}\langle\nabla r,\nabla r\rangle \geqslant -(m-1)H^2\big(1+r^2\big)^{-1} \quad \text{on } M \tag{2.46}$$

implies the estimate

$$\Delta r \leqslant (m-1)\frac{1+\sqrt{1+4H^2}}{2}r^{-1} \quad \text{as } r(x)\to+\infty \tag{2.47}$$

(see [1]). Proceeding in a way similar to that of the argument of Theorem 2.4, we prove

Theorem 2.5. *Let $(M,\langle\,,\,\rangle)$ be a complete manifold with Ricci tensor satisfying* (2.46). *Let $a(x)$ and $b(x)\in C^0(M)$ satisfy* (2.36) *and*

$$b(x) \leqslant Br(x)^{\beta} \quad \textit{for } r(x)\gg 1 \tag{2.48}$$

and some $B>0$, $\alpha<-2$, $\beta\leqslant\gamma(\sigma-1)-2$, with

$$\gamma > (m-1)\frac{1+\sqrt{1+4H^2}}{2} - 1. \tag{2.49}$$

Then any positive solution u of (2.39) *satisfies*

$$u(x) \geqslant Cr(x)^{-\gamma} \quad \textit{for } r(x)\gg 1 \tag{2.50}$$

and for some constant $C>0$.

Both the estimates of Theorems 2.4 and 2.5 are quite sharp with respect to the range of exponent γ. To simplify computations let us consider estimate (2.50). Towards this aim, for a chosen $R_0>1$, let M_g be the model with $g(r)\in C^\infty(M)$ positive on $[0,+\infty)$ and such that

$$g(r) = \begin{cases} r & \text{on } [0,\frac{1}{2}], \\ r^{\frac{1}{2}(1+\sqrt{1+4H^2})} & \text{on } [R_0,+\infty) \end{cases} \tag{2.51}$$

for some $H>0$. Since $Ricci_{(M,\langle\,,\,\rangle)}\langle\nabla r,\nabla r\rangle = -(m-1)\frac{g''(r)}{g(r)}$, we have

$$Ricci_{(M,\langle\,,\,\rangle)}\langle\nabla r,\nabla r\rangle = -(m-1)H^2r^{-2} \geqslant -(m-1)H^2\big(1+r^{-2}\big)\big(1+r^2\big)^{-1}$$

on $M_g\setminus B_{R_0}$. We let $\tilde{H}^2 = H^2(1+R_0^{-2})$ so that

$$Ricci_{(M,\langle\,,\,\rangle)}\langle\nabla r,\nabla r\rangle \geqslant -(m-1)\tilde{H}^2\big(1+r(x)^2\big)^{-1} \tag{2.52}$$

on $M_g\setminus B_{R_0}$. It is not hard to convince ourselves that $g(r)$ can be defined on $[\frac{1}{2},R_0]$ in such a way that (2.52) holds on all M_g. Now we set

$$\gamma = \frac{m-1}{2}\big(1+\sqrt{1+4\tilde{H}^2}\big) - 1$$

and we choose $A, B > 0$ sufficiently large that

$$\gamma(\gamma+2)R_0^2 - \gamma(m-1)\inf_{[0,R_0]} r\frac{g'(r)}{g(r)}\left(1+r^2\right)^{-\frac{\gamma}{2}-1} \\ \leqslant \gamma\left(1+R_0^2\right)^{-\frac{\gamma}{2}-1} + \left(1+R_0^2\right)^{-\frac{\gamma}{2}} A\inf_{[0,R_0]}(1+r)^{\alpha} + B\xi_0^{\sigma-1}\left(1+R_0^2\right)^{-\frac{\sigma\gamma}{2}}\inf_{[0,R_0]}(1+r)^{\beta} \tag{2.53}$$

for some $\xi_0 > 0$ fixed and $\alpha, \beta \in \mathbb{R}$. We set

$$a(x) = -A\left(1+r(x)\right)^{\alpha}, \qquad b(x) = B\left(1+r(x)\right)^{\beta}$$

and we define, with $\xi \geqslant \xi_0$,

$$v(x) = \xi\left(1+r(x)^2\right)^{-\frac{\gamma}{2}}. \tag{2.54}$$

Then, a simple computation shows that on M_g,

$$\begin{aligned}\Delta v + a(x)v - b(x)v^{\sigma} = \xi\bigg\{&\gamma(\gamma+2)r^2(x)\left(1+r^2(x)\right)^{-\frac{\gamma}{2}-2} - \gamma\left(1+r^2(x)\right)^{-\frac{\gamma}{2}-1} \\ &- \gamma(m-1)\frac{g'(r(x))}{g(r(x))}r(x)\left(1+r^2(x)\right)^{-\frac{\gamma}{2}-1} \\ &- A\left(1+r(x)\right)^{\alpha}\left(1+r^2(x)\right)^{-\frac{\gamma}{2}} \\ &- B\xi^{\sigma-1}\left(1+r(x)\right)^{\beta}\left(1+r(x)^2\right)^{-\frac{\sigma\gamma}{2}}\bigg\}.\end{aligned} \tag{2.55}$$

Thus, since $\xi \geqslant \xi_0$, (2.53) and inspection of (2.55) show that

$$\Delta v + a(x)v - b(x)v^{\sigma} \leqslant 0 \tag{2.56}$$

on B_{R_0}. Next, we note that on $M \setminus B_{R_0}$

$$(m-1)\frac{g'(r)}{g(r)} = \frac{m-1}{2}\frac{1}{r}\left(1+\sqrt{1+4H^2}\right) = (1+\gamma-\epsilon)\frac{1}{r} \tag{2.57}$$

for some $\epsilon = \epsilon(R_0) > 0$ (with $\epsilon \to 0$ as $R_0 \to +\infty$). Rearranging the terms in (2.55) and using (2.57) we have

$$\begin{aligned}\Delta v + a(x)v - b(x)v^{\sigma} = \xi r^2(x)\left(1+r^2(x)\right)^{-\frac{\gamma}{2}-2}\bigg\{&\epsilon\gamma - \gamma(\gamma+2-\epsilon)r(x)^{-2} \\ &- Ar(x)^{\alpha+2}\left(1+\frac{1}{r(x)}\right)^{\alpha}\left(1+\frac{1}{r^2(x)}\right)^{2} \\ &- B\xi^{\sigma-1}r(x)^{\beta+2-\gamma(\sigma-1)}\left(1+\frac{1}{r(x)}\right)^{\beta}\left(1+\frac{1}{r^2(x)}\right)^{2-\frac{\gamma}{2}(\sigma-1)}\bigg\}.\end{aligned} \tag{2.58}$$

If $\beta = (\sigma - 1)\gamma - 2$ we can choose $\xi \geqslant \xi_0$ sufficiently large that the above yields the validity of (2.56) on $M_g \setminus B_{R_0}$.

Note that in this example the range of α plays no role.

3. Uniqueness

The aim of this section is to prove a uniqueness result, which will be, as usual, an easy consequence of a companion comparison theorem. This latter is based on a form of the weak maximum principle as introduced in [12] and [14].

Theorem 3.1. *Let $a(x), b(x) \in C^0(M)$, $\sigma > 1$, $\tau \geqslant 0$, $\beta + \tau(\sigma - 1) > -2$ and suppose that $b(x) > 0$ on M,*

$$\text{(i)} \quad b(x) \geqslant B r(x)^\beta \quad \text{for } r(x) \gg 1 \quad \text{and} \quad \text{(ii)} \quad \sup_M \frac{a_-(x)}{b(x)} r(x)^{\tau(1-\sigma)} < +\infty. \tag{3.1}$$

Let $u, v \in C^2(M)$ be non-negative solutions of

$$\Delta u + a(x)u - b(x)u^\sigma \geqslant 0 \geqslant \Delta v + a(x)v - b(x)v^\sigma \tag{3.2}$$

on M, satisfying

$$\text{(i)} \quad v(x) \geqslant C_1 r(x)^\tau, \qquad \text{(ii)} \quad u(x) \leqslant C_2 r(x)^\tau \tag{3.3}$$

for $r(x) \gg 1$ and some positive constants C_1, C_2. If

$$\liminf_{R \to +\infty} \frac{\log vol\, B_R}{R^{2+\beta+\tau(\sigma-1)}} < +\infty \tag{3.4}$$

then $u(x) \leqslant v(x)$ on M.

Proof. First of all, let $u(x) \not\equiv 0$ otherwise there is nothing to prove. Next, we observe, by (3.2) and the strong maximum principle, that $v(x) > 0$ on M. This fact, $u(x) \not\equiv 0$ and (3.3)(i), (ii) imply that

$$\xi = \sup_M \frac{u(x)}{v(x)} \tag{3.5}$$

satisfies $0 < \xi < +\infty$. If $\xi \leqslant 1$, then $u(x) \leqslant v(x)$ on M. Let us assume, by contradiction, $\xi > 1$ and define

$$\phi = u - \xi v.$$

Note that $\phi \leqslant 0$ on M. We claim

$$\sup_M r(x)^{-\tau} \phi(x) = 0. \tag{3.6}$$

Indeed let $\{x_n\} \subset M$ be a sequence realizing ξ. Then

$$r(x_n)^{-\tau}\phi(x_n) = r(x_n)^{-\tau}v(x_n)\left\{\frac{u(x_n)}{v(x_n)} - \xi\right\}. \tag{3.7}$$

Now observe that

$$r(x_n)^{-\tau}v(x_n)$$

is bounded, because otherwise (3.3)(ii) would imply $\xi = 0$. From (3.7) it thus follows that $r(x_n)^{-\tau}\phi(x_n) \to 0$ as $n \to +\infty$ proving (3.6). We now use (3.2) to obtain

$$\Delta\phi \geqslant -a(x)\phi + b(x)\big(u^\sigma - (\xi v)^\sigma\big) + b(x)v^\sigma\xi\big(\xi^{\sigma-1} - 1\big). \tag{3.8}$$

By the mean value theorem we have

$$\big(u^\sigma - (\xi v)^\sigma\big)(x) = h(x)\phi(x), \tag{3.9}$$

where

$$h(x) = \begin{cases} \sigma u(x)^{\sigma-1} & \text{if } u(x) = \xi v(x), \\ \dfrac{\sigma}{u(x) - \xi v(x)} \displaystyle\int_{\xi v(x)}^{u(x)} t^{\sigma-1}\,dt & \text{if } u(x) \neq \xi v(x) \end{cases}$$

is continuous and non-negative on M. Furthermore, by the mean value theorem for integrals, for some $y \in (\xi v(x), u(x))$ or $y \in (u(x), \xi v(x))$,

$$\begin{aligned} h(x) &= \frac{\sigma}{u(x) - \xi v(x)}\big(u(x) - \xi v(x)\big)y^{\sigma-1} \\ &\leqslant \sigma\big[u^{\sigma-1}(x) + \big(\xi v(x)\big)^{\sigma-1}\big] \\ &\leqslant Cr(x)^{\tau(\sigma-1)} \end{aligned} \tag{3.10}$$

for some appropriate $C > 0$ because, as we have already observed, $r(x)^{-\tau}v(x)$ is bounded above due to (3.3)(ii) and $\xi > 0$. Note that, since $b(x) > 0$ on M, we can rewrite (3.1)(i) as

$$b(x) \geqslant \tilde{B}\big(1 + r(x)\big)^\beta \quad \text{on } M \tag{3.11}$$

for some appropriate $\tilde{B} > 0$. Using $b(x) > 0$ and (3.9), from (3.8) we deduce

$$\frac{1}{b(x)}\Delta\phi \geqslant \left(\frac{a_-(x)}{b(x)} + h(x)\right)\phi(x) + v^\sigma(x)\xi\big(\xi^{\sigma-1} - 1\big)$$

and therefore, from $\xi > 1$, (3.1)(ii), (3.10), (3.3)(i) and $\phi(x) \leqslant 0$

$$\big(1 + r(x)\big)^{-\sigma\tau}\frac{1}{b(x)}\Delta\phi \geqslant C\big(1 + r(x)\big)^{-\tau}\phi(x) + D\xi\big(\xi^{\sigma-1} - 1\big) \quad \text{on } M$$

for some appropriate constants $C, D > 0$. Next we choose $\epsilon > 0$ sufficiently small, so that

$$C\big(1+r(x)\big)^{-\tau}\phi(x) \geqslant -\frac{1}{2}D\xi\big(\xi^{\sigma-1}-1\big) \tag{3.12}$$

on

$$\Omega_\epsilon = \big\{x \in M\colon\ \phi(x) > -\epsilon\big\}.$$

Note that this is possible since $\tau \geqslant 0$. Then on Ω_ϵ, $\Delta\phi \geqslant 0$ so that (3.11) implies

$$\big(1+r(x)\big)^{-\beta-\sigma\tau}\Delta\phi \geqslant \big(1+r(x)\big)^{-\sigma\tau}\frac{1}{b(x)}\Delta\phi$$

and thus

$$\inf_{\Omega_\epsilon}\big(1+r(x)\big)^{-\beta-\sigma\tau}\Delta\phi \geqslant \frac{1}{2}D\xi\big(\xi^{\sigma-1}-1\big) > 0 \tag{3.13}$$

since $\xi > 1$. This fact, together with (3.4) contradicts the theorem on p. 58 of [11]. □

We observe that, since $\tau \geqslant 0$, assumption (3.3)(i) implies

$$\liminf_{r(x)\to+\infty} v(x) > 0.$$

This fact is essential for the validity of the theorem. Indeed, consider $\mathbb{H}^m$ the hyperbolic space of constant sectional curvature -1 and dimension m. Let $m \geqslant 3$. On $\mathbb{H}^m$ the equation

$$\Delta u + \frac{m(m-2)}{4}u - u^{\frac{m+2}{m-2}} = 0$$

admits a family of positive solutions $w_a(x)$, $a > 1$, given by

$$w_a(x) = \frac{1}{m(m-2)a^2}\left(a^2 - \tanh^2\frac{r(x)}{2}\right)^{-\frac{m-2}{2}}\left(2\cosh^2\frac{r(x)}{2}\right)^{-\frac{m-2}{2}}.$$

Here we have realized $\mathbb{H}^m$ as $\mathbb{R}^m$, with metric in polar coordinates on $(0,+\infty)\times\ S^{m-1}$

$$\langle\,,\rangle = dr^2 + \sinh^2 r\, d\theta^2,$$

$d\theta^2$ the standard metric on the unit sphere S^{m-1}. Note that

$$w_a(x) \sim Ce^{-\frac{m-2}{2}r(x)} \quad \text{as } r(x) \to +\infty \tag{3.14}$$

for some appropriate constant $C > 0$. In particular $w_a(x) \to 0$ as $r(x) \to +\infty$.

Chosen two different values, say a_1 and a_2, for the parameter, we set $u = w_{a_1}$, $v = w_{a_2}$. Then

$$\frac{v(x)}{u(x)} \sim \left(\frac{a_2^2-1}{a_1^2-1}\right)^{-\frac{m-2}{2}}\left(\frac{a_1}{a_2}\right)^2 \quad \text{as } r(x) \to +\infty$$

and we can choose a_1, a_2 so that

$$\left(\frac{a_2^2-1}{a_1^2-1}\right)^{-\frac{m-2}{2}}\left(\frac{a_1}{a_2}\right)^2 > 1.$$

In this case $v(x) > u(x)$ somewhere on $\mathbb{H}^m$. Furthermore (3.2) is satisfied with $a(x) \equiv \frac{m(m-2)}{4}$, $b(x) \equiv 1$, $\sigma = \frac{m+2}{m-2}$ while (3.1)(i) holds with $\beta = 0$ and (3.1)(ii) with any τ since $a_-(x) \equiv 0$. Furthermore,

$$\liminf_{R\to+\infty} \frac{\log vol\, B_R}{R^2} = 0 \tag{3.15}$$

and we think of (3.3)(ii) to be met with $\tau = 0$, for instance. In this case all of the assumptions, but (3.3)(i), of Theorem 3.1 are satisfied.

As an immediate consequence of Theorem 3.1 we obtain the following uniqueness result

Corollary 3.2. *In the assumption of Theorem* 3.1 *suppose that $u(x), v(x) \in C^2(M)$ are two non-negative solutions of*

$$\Delta u + a(x)u - b(x)u^\sigma = 0 \quad \text{on } M$$

satisfying

$$C^{-1} r(x)^\tau \leqslant u(x), v(x) \leqslant C r(x)^\tau \tag{3.16}$$

for $r(x) \gg 1$ and some constant $C > 0$. Suppose also the validity of (3.4). *Then*

$$u \equiv v.$$

Putting together Corollary 3.2 and the "a priori" estimates of Section 1 we have

Theorem 3.3. *Let $(M, \langle\,,\rangle)$ be a complete manifold with Ricci tensor satisfying*

$$Ricci_{(M,\langle\,,\rangle)} \geqslant -(m-1)H^2\left(1+r^2\right)^{\frac{\delta}{2}} \tag{3.17}$$

for some $H > 0$, $\delta \geqslant -2$. Let $a(x), b(x) \in C^0(M)$ with $b(x) > 0$ on M and assume that, for some $A, B > 0$,

$$A^{-1} r(x)^\alpha \leqslant a(x) \leqslant A r(x)^\alpha, \tag{3.18}$$

$$B^{-1} r(x)^\beta \leqslant b(x) \leqslant B r(x)^\beta \tag{3.19}$$

for $r(x) \gg 1$ and with $\alpha > \delta$, $\alpha \geqslant \max\{\beta, \beta + \frac{\delta}{2} - 1\}$. Then there exists at most one non-negative, non-trivial solution $u \in C^2(M)$ of

$$\Delta u + a(x)u - b(x)u^\sigma = 0, \quad \sigma > 1, \quad \text{on } M. \tag{3.20}$$

Proof. First observe, by the maximum principle, that if $u \geqslant 0$, $u \not\equiv 0$ is a solution of (3.20) then $u > 0$ on M. Thus if $u \not\equiv 0$, according to the "a priori" estimates of Theorems 2.1 and 2.3 we have (note that $\alpha \geqslant \frac{\delta}{2} - 1$ because $\alpha > \delta \geqslant -2$, while $\beta \leqslant 1 - \frac{\delta}{2} + \alpha$ because of our assumptions)

$$C^{-1} r(x)^{-\frac{\beta-\alpha}{\sigma-1}} \leqslant u(x) \leqslant C r(x)^{-\frac{\beta-\alpha}{\sigma-1}} \tag{3.21}$$

for some constant $C > 0$ and $r(x) \gg 1$. To apply Corollary 3.2, since $a_-(x) \equiv 0$ for $r(x) \gg 1$ because of (3.18), we only need to have $\alpha \geqslant \beta$ and $\alpha > -2$. This latter is guaranteed by $\alpha > \delta$, while the first is satisfied by assumption. It remains to check that (3.4) holds, that is,

$$\liminf_{R \to +\infty} \frac{\log vol\, B_R}{R^{\alpha+2}} < +\infty. \tag{3.22}$$

Towards this aim, we need to estimate $vol\, B_R$ from above using assumption (3.17). As reported in [11, p. 33], if $\delta \geqslant 0$, we have

$$\log vol\, B_R \leqslant C R^{1+\frac{\delta}{2}} \quad \text{as } R \to +\infty$$

for some constant $C > 0$. In this case (3.22) is satisfied if

$$\alpha \geqslant \frac{\delta}{2} - 1. \tag{3.23}$$

If $-2 < \delta < 0$ then

$$vol\, B_R \leqslant C \int_0^R s^{-\frac{\delta}{2}(m-1)} e^{\frac{2H}{2+\delta}(1+s)^{1+\frac{\delta}{2}}(m-1)}\, ds$$

for some $C > 0$. Thus

$$\frac{\log vol\, B_R}{R^{\alpha+2}} \asymp \frac{1}{R^{\alpha+2}} \quad \text{as } R \to +\infty$$

and (3.22) is satisfied if

$$\alpha \geqslant -2. \tag{3.24}$$

If $\delta = -2$ then

$$vol\, B_R \leqslant C \int_0^R s^{\frac{1+\sqrt{1+4H^2}}{2}(m-1)}\, ds \asymp R^{\frac{1+\sqrt{1+4H^2}}{2}(m-1)+1}$$

hence (3.22) is satisfied if

$$\alpha > -2. \tag{3.25}$$

Note that $\alpha > \delta$ implies (3.23), (3.24) and (3.25). $\square$

We underline that even in case of Euclidean space $\mathbb{R}^m$, Theorem 3.3 generalizes recent results. For instance, Theorem 8 of [5], where positive solutions have to be taken in a special class H (see [5] for its definition) and stronger assumptions have to be required on the coefficients $a(x)$ and $b(x)$; or it generalizes Theorem 1.2 of [3], always on $\mathbb{R}^m$, where one needs to assign the exact asymptotic behaviour of $a(x)$ and $b(x)$ at infinity.

In case of Yamabe equation, that is, when (3.20) takes the form (for $m \geqslant 3$)

$$\frac{4(m-1)}{m-2}\Delta u(x) - s(x)u(x) + K(x)u(x)^{\frac{m+2}{m-2}} = 0, \quad u > 0, \tag{3.26}$$

with $s(x)$ the scalar curvature of $(M, \langle\,,\rangle)$ and $K(x)$ the assigned scalar curvature of the conformally deformed metric

$$g(\,,) = u^{\frac{4}{m-2}}\langle\,,\rangle, \tag{3.27}$$

due to the particular geometric properties of (3.26) we can give the following uniqueness result avoiding bounding the Ricci curvature from below.

Theorem 3.4. *Let $(M, \langle\,,\rangle)$ be a complete manifold of dimension $m \geqslant 3$ and scalar curvature $s(x)$ satisfying*

$$\sup_M s_+(x) < +\infty. \tag{3.28}$$

Let $K(x) \in C^\infty(M)$, $K(x) < 0$ on M and suppose that

$$K(x) \leqslant -\frac{B}{r(x)^\beta} \quad \textit{for } r(x) \gg 1, \tag{3.29}$$

for some constants $B > 0$, $\beta < 2$ and

$$\inf_M \frac{s_+(x)}{K(x)} > -\infty. \tag{3.30}$$

Assume

$$\liminf_{R\to+\infty} \frac{\log vol\, B_R}{R^{2-\beta}} < +\infty. \tag{3.31}$$

Then there exists at most one conformal deformation to a new metric g as in (3.27) *with scalar curvature $K(x)$ and such that $u_* = \inf_M u > 0$.*

Proof. Suppose such a metric exists. Then $u > 0$ has to satisfy (3.26). In the assumptions (3.28)–(3.31) we can apply Theorem B of [10] to deduce that $u^* = \sup_M u < +\infty$. Let $r_g(x) = dist_{(M,g)}(x, o)$. Then, using $u_* > 0$, we can conclude that there exists a constant $C > 0$ such that

$$C^{-1}r(x) \leqslant r_g(x) \leqslant Cr(x) \quad \forall x \in M,$$
$$d\,vol_g \leqslant C^m d\,vol_{\langle\,,\rangle} \tag{3.32}$$

where $d\,vol_g$ and $d\,vol_{\langle\,,\,\rangle}$ are the elements of volume respectively in the metrics g and $\langle\,,\,\rangle$. Thus

$$B_R^g = \{x \in M\colon\ r_g(x) < R\} \subseteq \{y \in M\colon\ r(y) < CR\} = B_{CR}, \tag{3.33}$$

and using the above and (3.31) we deduce

$$\liminf_{R\to+\infty} \frac{\log vol\, B_R^g}{R^{2-\beta}} \leqslant C^{2-\beta} \liminf_{R\to+\infty} \frac{\log vol\, B_{CR}}{(CR)^{2-\beta}} < +\infty. \tag{3.34}$$

Let now $\tilde{g} = \tilde{u}^{\frac{4}{m-2}} \langle\,,\,\rangle$ be a second metric with the same properties of g. Then

$$\tilde{g} = w^{\frac{4}{m-2}} g$$

with $w = \frac{\tilde{u}}{u}$ and, since g and $\tilde{g}$ have the same scalar curvature $K(x)$, w has to satisfies

$$\frac{4(m-1)}{m-2} \Delta_g w - K(x)w + K(x)w^{\frac{m+2}{m-2}} = 0 \tag{3.35}$$

on M, where Δ_g is the Laplace–Beltrami operator with respect to the complete metric g on M. We rewrite the above as

$$\Delta_g w = -\frac{m-2}{4(m-1)} K(x)\{w^{\frac{m+2}{m-2}} - w\} \tag{3.36}$$

and we note that

$$w^* = \sup_M w \leqslant \frac{\tilde{u}^*}{u_*} < +\infty. \tag{3.37}$$

We shall now show that $w^* \leqslant 1$. Towards this aim we reason by contradiction and suppose that $w^* > 1$. First of all we observe that, since $K(x) < 0$ on M, using (3.32) we can rewrite (3.29) in the form

$$K(x) \leqslant -\frac{A}{(1 + r_g(x))^\beta} \tag{3.38}$$

on M for some appropriate constant $A > 0$. Next we fix $1 < \gamma < w^*$ and let

$$\Omega_\gamma = \{x \in M\colon\ w(x) > \gamma\} \neq \emptyset.$$

From (3.36) and $K(x) < 0$, it follows that

$$\Delta_g w \geqslant 0 \quad \text{on } \Omega_\gamma.$$

Thus, using (3.38) and (3.36) we deduce

$$(1 + r_g)^\beta \Delta_g w \geqslant A\{w^{\frac{m+2}{m-2}} - w\} > A\gamma\{\gamma^{\frac{4}{m-2}} - 1\} > 0 \quad \text{on } \Omega_\gamma. \tag{3.39}$$

On the other hand, because of (3.34), we can apply the theorem on [11, p. 58] with $\sigma = 0$ because of (3.37), to the manifold (M, g) to deduce that

$$\inf_{\Omega_\gamma}(1 + r_g)^\beta \Delta_g w \leqslant 0 \tag{3.40}$$

contradicting (3.39). Hence $w^* \leqslant 1$, that is, $\tilde{u} \leqslant u$ on M. Inverting the rôle of g and $\tilde{g}$ we conclude $\tilde{u} \equiv u$ on M. $\square$

Remark 3.1. In case $m = 2$, Yamabe equation takes the form

$$\Delta u - s(x) + K(x)e^{2u} = 0, \tag{3.41}$$

with $s(x)$ the Gaussian curvature of $(M, \langle\,,\rangle)$ and $K(x)$ the assigned Gaussian curvature of the conformally deformed metric

$$g(\,,) = e^{2u}\langle\,,\rangle,$$

where u, in this case, is not necessarily positive.

If we suppose

$$u_* = \inf_M u > -\infty, \tag{3.42}$$

the conclusion of Theorem 3.4 continues to hold under the same assumptions. This can be seen by modifying the above proof as follows.

Set

$$v = e^u,$$

then, using (3.41), v satisfies

$$\Delta v - s(x)v + K(x)v^3 \geqslant 0.$$

Applying Theorem B of [10], we deduce $\sup_M v < +\infty$ and in turn $u^* = \sup_M u < +\infty$. We observe that (3.42) guarantees the validity of (3.32) and thus that of (3.34). Next, if u and $\tilde{u}$ are solutions of (3.41), the function

$$w = u - \tilde{u}$$

must satisfy

$$\Delta w - K(x) + K(x)e^{2w} = 0 \tag{3.43}$$

since $g(\,,) = e^{2u}\langle\,,\rangle$ and $\tilde{g}(\,,) = e^{2\tilde{u}}\langle\,,\rangle$ have the same Gaussian curvature $K(x)$. The proof now follows the same lines of that of case $m \geqslant 3$.

4. Existence

In this section we apply the method of sub- and super-solutions to provide existence for the problem

$$\begin{cases} \text{(i)} \quad \Delta u + a(x)u - b(x)u^{\sigma} = 0, \quad \sigma > 1, \\ \text{(ii)} \quad u > 0 \quad \text{on } M \end{cases} \tag{4.1}$$

under appropriate assumptions on the coefficients $a(x)$ and $b(x)$. We begin with a few introductory definitions.

Let $L = \Delta + a(x)$, $a(x) \in C^0(M)$ and let Ω be a non-empty open subset of M with smooth boundary $\partial\Omega$; the first eigenvalue $\lambda_1(\Omega)$ of the operator L on Ω is given by the variational characterization

$$\lambda_1^L(\Omega) = \inf \frac{\int_\Omega |\nabla\phi|^2 - a(x)\phi^2}{\int_\Omega \phi^2} \tag{4.2}$$

where the infimum is taken over $\phi \in C_0^\infty(\Omega)$, $\phi \not\equiv 0$. In fact there exists a unique eigenfunction v defined on $\bar{\Omega}$, so that

$$\begin{cases} \Delta v + a(x)v + \lambda_1^L(\Omega)v = 0 \quad \text{on } \Omega, \\ v > 0 \quad \text{on } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

For an arbitrary bounded subset S of M we define the first eigenvalue $\lambda_1^L(S)$ by an approximation procedure, that is

Definition 4.1. Let $S \subset M$ be an arbitrary bounded subset of M. The first eigenvalue of the operator L on S, $\lambda_1^L(S)$ is defined by

$$\lambda_1^L(S) = \sup \lambda_1^L(\Omega), \tag{4.4}$$

where the supremum is taken over all open sets Ω with smooth boundary such that $S \subset \Omega$. In particular, if $S = \emptyset$, then $\lambda_1^L(S) = +\infty$.

Then,

Theorem 4.1. *(See [12].) Let $a(x), b(x) \in C^{0,\mu}_{\mathrm{loc}}(M)$ for some $0 < \mu \leqslant 1$ and suppose that $b(x) \geqslant 0$ on M and that the set*

$$B_0 = \{x \in M\colon\ b(x) = 0\}$$

is bounded. Suppose furthermore that $\lambda_1^L(B_0) > 0$. If $u_- \in C^0 \cap H^1_{\mathrm{loc}}(M)$, $u \geqslant 0$, $u \not\equiv 0$ is a global subsolution of (4.1)(i) *then* (4.1) *has a maximal positive C^2-solution.*

We recall that a positive solution u of (4.1) is said to be *maximal* if for any other positive solution v we have $v \leqslant u$ on M.

We are now ready to prove the main result of this section.

Theorem 4.2. *Let $(M, \langle\,,\rangle)$ be a complete manifold; let $a(x), b(x) \in C^{0,\mu}(M)$ for some $0 < \mu \leqslant 1$ and suppose that $b(x) \geqslant 0$ and strictly positive outside the ball B_R. Assume that the Ricci tensor satisfies*

$$Ricci_{(M,\langle\,,\rangle)} \geqslant -(m-1)H^2\big(1+r(x)^2\big)^{\frac{\delta}{2}} \quad \text{on } M \setminus B_R \tag{4.5}$$

for some $\delta \geqslant -2$ and $H > 0$. Let $B_0 = \{x \in M\colon\ b(x) = 0\}$ and assume that

$$\lambda_1^L(B_0) > 0 \tag{4.6}$$

and

$$\text{(i)}\quad a(x) \geqslant Ar^{\alpha}(x), \qquad \text{(ii)}\quad b(x) \leqslant Br^{\beta}(x)e^{Dr^{\theta}(x)} \tag{4.7}$$

on $M \setminus B_R$, for some $\alpha > \frac{\delta}{2} - 1$, $\theta < \min\{1 + \alpha - \frac{\delta}{2}, 1 + \frac{\alpha}{2}\}$, and $A, B > 0$, $D, \beta \in \mathbb{R}$. Moreover, let

$$\tau = \inf_{B_R} r\Delta r \quad (\text{in the weak sense}) \tag{4.8}$$

and suppose

$$a(x) > \begin{cases} -\dfrac{1+\tau}{4R^2} & \text{if } \tau \geqslant 0, \\ -\dfrac{(1+\tau)^2}{4R^2} & \text{if } -1 \leqslant \tau < 0, \end{cases} \tag{4.9}$$

$$a(x) \geqslant 0 \quad \text{if } \tau < -1 \tag{4.10}$$

on B_R. Then problem (4.1) *admits a maximal positive C^2-solution on M.*

Remark 4.1. If $R > 0$ is so small that B_R is a regular ball, then pointwise we have

$$\Delta r \geqslant (m-1)r^{-1}\big(1 + o(1)\big) \quad \text{as } r \to 0^+.$$

Thus up to choosing R small, $\tau \geqslant 0$ on B_R. In this case Theorem 4.2 holds simply requiring

$$a(x) > -\frac{1+\tau}{4R^2} \quad \text{on } B_R. \tag{4.11}$$

In particular negativity of $a(x)$ is always allowed in a sufficiently small geodesic ball.

Remark 4.2. Consider for instance case $\delta = 0$ with the geometry of M controlled from above by the geometry of hyperbolic space of constant negative curvature $-H^2$. The condition on α becomes $\alpha > -1$, so that a behaviour of $a(x) \geqslant 0$ of the type $Ar(x)^{-1+\epsilon} \leqslant a(x) \leqslant C(1 + r(x))^{-1+2\epsilon}$, for $r(x) \gg 1$, some $\epsilon > 0$, is admissible. In this case, for $C > 0$ sufficiently small, $\lambda_1^L(M) \geqslant 0$, $L = \Delta + a(x)$. This shows that Theorem 4.2 is not contained in Theorem 2.1 of [1], where $\lambda_1^L(M) < 0$.

Remark 4.3. Given $L = \Delta + a(x)$, conditions on the sign of the spectral radius $\lambda_1^L(M)$ are related to the principal eigenvalue λ_* for the equation $\Delta u + \lambda a(x)u = 0$. This is reported in [12]. Existence results on $\mathbb{R}^m$ are often stated in the literature under conditions on λ_*. For instance the existence part in Theorem 1.2 of [4] corresponds to $\lambda_* = 0$ and in this case one has $\lambda_1^L(M) < 0$. See [12] for a detailed discussion.

Remark 4.4. The upper bound for θ is sharp in the sense that for equality we have examples of non existence (see [1, Theorem 3.2]).

Proof. According to Theorem 4.1 we only need to construct a global subsolution $u_- \geqslant 0$, $u_- \not\equiv 0$. It is clear that, without loss of generality, we can limit ourselves to consider the case $D, \theta > 0$.

Step 1. Construction of a subsolution inside B_R.

Recall that $B_0 \subset B_R$. We look for a subsolution v of the form $v(x) = \gamma(r(x))$ with $\gamma : [0, R] \to \mathbb{R}^+$ satisfying

$$\begin{cases} \gamma'' + \Delta r \gamma' + \tilde{A}\gamma - \tilde{B}\gamma^\sigma \geqslant 0, \\ \gamma > 0, \quad \gamma'(0) = 0, \end{cases} \tag{4.12}$$

where

$$\tilde{A} = \min_{\overline{B_R}}\big(a(x)\big) \quad \text{and} \quad \tilde{B} = \max_{\overline{B_R}}\big(b(x)\big) > 0.$$

We define

$$\gamma(r) = \gamma_0\big(\zeta r^2 + 1\big)^{\frac{1}{2}} \tag{4.13}$$

with $\gamma_0, \zeta > 0$ to be determined.

Since

$$\gamma'(r) = \gamma_0 \zeta r\big(\zeta r^2 + 1\big)^{\frac{1}{2}-1}$$

and

$$\gamma''(r) = \gamma_0 \zeta \big(\zeta r^2 + 1\big)^{\frac{1}{2}-2},$$

we obtain that γ is a solution of (4.12) if and only if

$$\begin{aligned} \gamma''(r) + \Delta r \gamma'(r) &= \gamma_0 \zeta \big(\zeta r^2 + 1\big)^{\frac{1}{2}-2}\big\{1 + r\Delta r\big(\zeta r^2 + 1\big)\big\} \\ &\geqslant \tilde{B}\gamma_0^\sigma \big(\zeta r^2 + 1\big)^{\frac{\sigma}{2}} - \tilde{A}\gamma_0\big(\zeta r^2 + 1\big)^{\frac{1}{2}}. \end{aligned} \tag{4.14}$$

Inequality (4.14) is equivalent to

$$\zeta\big\{1 + r\Delta r\big(\zeta r^2 + 1\big)\big\} \geqslant \tilde{B}\gamma_0^{\sigma-1}\big(\zeta r^2 + 1\big)^{\frac{\sigma}{2}+\frac{3}{2}} - \tilde{A}\big(\zeta r^2 + 1\big)^2. \tag{4.15}$$

Now we observe that if $\tilde{A} \geqslant 0$

$$\begin{aligned}\zeta\{1 + r\Delta r(\zeta r^2 + 1)\} &\geqslant \zeta\{1 + \tau(\zeta r^2 + 1)\} \\ &\geqslant \tilde{B}\gamma_0^{\sigma-1}(\zeta R^2 + 1)^{\frac{\sigma}{2}+\frac{3}{2}} - \tilde{A} \\ &\geqslant \tilde{B}\gamma_0^{\sigma-1}(\zeta R^2 + 1)^{\frac{\sigma}{2}+\frac{3}{2}} - \tilde{A}(\zeta R^2 + 1)^2.\end{aligned}$$

Thus, if $\tilde{A} \geqslant 0$, it is enough to choose $\gamma_0 \gg 1$ and γ is a subsolution.

If $\tilde{A} < 0$, γ is a subsolution if there exist γ_0 and ζ such that

$$\zeta\{1 + \tau(\zeta r^2 + 1)\} \geqslant \tilde{B}\gamma_0^{\sigma-1}(\zeta R^2 + 1)^{\frac{\sigma}{2}+\frac{3}{2}} - \tilde{A}(\zeta R^2 + 1)^2. \tag{4.16}$$

Towards this end we consider the two cases: $\tau \geqslant 0$ and $\tau < 0$.

1. If $\tau \geqslant 0$

$$\zeta\{1 + \tau(\zeta r^2 + 1)\} \geqslant \zeta(1 + \tau)$$

and so if there exists $\zeta > 0$ such that

$$\zeta(1 + \tau) > -\tilde{A}(\zeta R^2 + 1)^2 \tag{4.17}$$

(note that $-\tilde{A} > 0$), we can choose γ_0 sufficiently small in such the way that

$$\zeta(1 + \tau) > \tilde{B}\gamma_0^{\sigma-1}(\zeta R^2 + 1)^{\frac{\sigma}{2}+\frac{3}{2}} - \tilde{A}(\zeta R^2 + 1)^2$$

and therefore γ is a subsolution.

Since $\tilde{A} < 0$, inequality (4.17) admits positive solutions (in the ζ variable) if the quadratic equation

$$\tilde{A}R^4\zeta^2 + (2\tilde{A}R^2 + (1 + \tau))\zeta + \tilde{A} = 0 \tag{4.18}$$

has positive discriminant, and at least one positive root. This is the case when

$$\tilde{A} > -\frac{1 + \tau}{4R^2}. \tag{4.19}$$

2. If $\tau < 0$,

$$\zeta\{1 + \tau(\zeta r^2 + 1)\} \geqslant \zeta\{1 + \tau(\zeta R^2 + 1)\}$$

and the request becomes

$$R^2(\tau + \tilde{A}R^2)\zeta^2 + (1 + \tau + 2\tilde{A}R^2)\zeta + \tilde{A} > 0.$$

Analogously to what we did above, this inequality admits positive solutions if

$$\tilde{A} > -\frac{(1+\tau)^2}{4R^2} \quad \text{if } -1 \leqslant \tau < 0$$

or

$$\tilde{A} \geqslant 0 \quad \text{if } \tau < -1.$$

Step 2. Construction of a subsolution on $M \setminus B_R$.

We observe that, because of (4.5), (4.7) and the Laplacian comparison theorem a non-negative, non-increasing function w satisfying on $[R, +\infty)$

$$w'' + (m-1)\tilde{H} r^{\frac{\delta}{2}}\big(1+o(1)\big)w' + Ar^{\alpha} w - Br^{\beta} e^{Dr^{\theta}} w^{\sigma} \geqslant 0 \tag{4.20}$$

($o(1)$ as $r \to +\infty$) where

$$\tilde{H} = \begin{cases} H & \text{if } \delta > -2, \\ \dfrac{1+\sqrt{1+4H^2}}{2} & \text{if } \delta = -2 \end{cases}$$

gives rise to a subsolution $w_-(x) = w(r(x))$ of (4.1) on $M \setminus B_R$. We look for a function w of the form

$$w = \big(\mu + e^{Dr^{\theta}}\big)^{\xi}, \quad \xi < \frac{1}{1-\sigma} < 0, \tag{4.21}$$

where $\mu > 0$ is a sufficiently large positive constant and $\theta < \min\{1 + \alpha - \frac{\delta}{2}, 1 + \frac{\alpha}{2}\}$.

We now prove that $w(r)$, chosen as in (4.21), is a solution of (4.20). Observe that

$$w' = \theta \xi D\big(\mu + e^{Dr^{\theta}}\big)^{\xi-1} e^{Dr^{\theta}} r^{\theta-1} < 0$$

and

$$\begin{aligned} w'' &= \theta^2 \xi(\xi-1) D^2\big(\mu + e^{Dr^{\theta}}\big)^{\xi-2} e^{2Dr^{\theta}} r^{2(\theta-1)} + \theta^2 \xi D^2\big(\mu + e^{Dr^{\theta}}\big)^{\xi-1} e^{Dr^{\theta}} r^{2(\theta-1)} \\ &\quad + \theta(\theta-1)\xi D\big(\mu + e^{Dr^{\theta}}\big)^{\xi-1} e^{Dr^{\theta}} r^{\theta-2}. \end{aligned}$$

Now define

$$\begin{aligned} H_w(r) &= w'' + (m-1)\tilde{H} r^{\frac{\delta}{2}}\big(1+o(1)\big)w' + Ar^{\alpha} w \\ &= D^2\theta^2\xi(\xi-1)\big(\mu + e^{Dr^{\theta}}\big)^{\xi-2} e^{2Dr^{\theta}} r^{2(\theta-1)} + D^2\theta^2\xi\big(\mu + e^{Dr^{\theta}}\big)^{\xi-1} e^{Dr^{\theta}} r^{2(\theta-1)} \\ &\quad + \theta(\theta-1) D\xi\big(\mu + e^{Dr^{\theta}}\big)^{\xi-1} e^{Dr^{\theta}} r^{\theta-2} \\ &\quad + D\theta\xi(m-1)\tilde{H} r^{\frac{\delta}{2}}\big(1+o(1)\big)\big(\mu + e^{Dr^{\theta}}\big)^{\xi-1} e^{Dr^{\theta}} r^{\theta-1} \\ &\quad + Ar^{\alpha}\big(\mu + e^{Dr^{\theta}}\big)^{\xi}. \end{aligned} \tag{4.22}$$

Using the assumptions on α, θ, which in particular imply $\alpha > -2$ and $\theta < \alpha + 2$, we have

$$H_w(r) \sim Ar^{\alpha}\big(\mu + e^{Dr^{\theta}}\big)^{\xi} \quad \text{as } r \to +\infty$$

and

$$H_w(r) > 0 \quad \text{on } [R, +\infty)$$

for μ sufficiently large, if necessary. Now w is a subsolution (for large r) if and only if

$$Br^{\beta}e^{Dr^{\theta}}\big(\mu + e^{Dr^{\theta}}\big)^{\xi\sigma} \leqslant Ar^{\alpha}\big(\mu + e^{Dr^{\theta}}\big)^{\xi},$$

in other words,

$$Ar^{\alpha-\beta}e^{-Dr^{\theta}}\big(\mu + e^{Dr^{\theta}}\big)^{\xi(1-\sigma)} \geqslant B, \quad \text{for } r \gg 1 \tag{4.23}$$

which is satisfied since $\xi < \frac{1}{1-\sigma}$ and $\theta > 0$. Thus w satisfies (4.20) on $[R, +\infty)$ up to choosing $\mu > 0$ large enough.

Step 3. Construction of a global subsolution.

For $u \in C^0(M) \cap H_2^1(M)$ we define

$$b_u = u^{-\sigma}\big\{\Delta u - a(x)u\big\} \quad \text{in the weak sense.} \tag{4.24}$$

We note that, for any constant $E > 0$,

$$b_{Eu} = E^{1-\sigma}b_u. \tag{4.25}$$

We also observe that u is a subsolution of (4.1) if and only if

$$b_u(x) \geqslant b(x), \quad \text{on } M. \tag{4.26}$$

Choose ϵ so small as to have

$$b(x) > 0 \quad \text{on } M \setminus B_{R-\epsilon} \tag{4.27}$$

(note that this is possible since $B_0 \Subset B_R$). Now let $\tilde{u}_-$ be a positive function in $C^0(M) \cap H_2^1(M)$ such that

$$\tilde{u}_-(x) = \begin{cases} v(x) & \text{if } x \in B_{R-\epsilon}, \\ w_-(x) & \text{if } x \in M \setminus B_R. \end{cases} \tag{4.28}$$

Next using (4.27), (4.25) and $\sigma > 1$ we choose $E > 0$ sufficiently small that $u_- = E\tilde{u}_-$ satisfies

$$b_{u_-} \geqslant b(x) \quad \text{on } M \setminus B_{R-\epsilon}.$$

Then $u_-(x)$ is a global subsolution of (4.1), with $u \not\equiv 0$. $\square$

Applying Theorem 4.2 to Yamabe equation, we obtain the following existence result.

Theorem 4.3. *Let $(M, \langle\,,\rangle)$ be a complete manifold of dimension $m \geqslant 3$ and scalar curvature $s(x)$. Let $K(x) \in C^\infty(M)$ such that $K(x) \leqslant 0$ and strictly negative outside B_R. If $K_0 = \{x \in M\colon\ K(x) = 0\} \subset B_R$, assume that the Ricci tensor of M satisfies*

$$Ricci_{(M,\langle\,,\rangle)} \geqslant -(m-1)H^2\big(1 + r(x)^2\big)^{\frac{\delta}{2}} \quad on\ M \setminus B_R \tag{4.29}$$

for some $\delta > -2$ and $H > 0$. Moreover, assume that

$$\lambda_1^L(K_0) > 0 \tag{4.30}$$

where $L = c_m\Delta - s(x)$, $c_m = \frac{4(m-1)}{m-2}$, and

$$\text{(i)}\quad s(x) \leqslant -Ar^\alpha(x), \qquad \text{(ii)}\quad K(x) \geqslant -Br^\beta(x)e^{Dr^\theta(x)} \tag{4.31}$$

on $M \setminus B_R$, for some $\frac{\delta}{2} - 1 < \alpha \leqslant \delta$, $\theta < \min\{1 + \alpha - \frac{\delta}{2}, 1 + \frac{\alpha}{2}\}$, and $\beta, D \in \mathbb{R}$, $A, B > 0$. Then there exists $\eta > 0$ such that, if

$$s(x) \leqslant \eta \quad on\ M, \tag{4.32}$$

then the metric $\langle\,,\rangle$ can be conformally deformed to a metric $(\,,)$ with scalar curvature $K(x)$.

Remark 4.5. Since $s(x)$ is the trace of Ricci tensor we must necessarily have $A \leqslant m(m-1)H^2$.

Proof. First of all observe that it is enough to prove the result for $D > 0$, $0 < \theta < \min\{1 + \alpha - \frac{\delta}{2}, 1 + \frac{\alpha}{2}, \alpha + 2\}$. Next we need the following result in [13] which depends on the geometrical properties of Yamabe equation.

Proposition 4.4. *Let $\Omega_0 \Subset \Omega_1 \subset (M, \langle\,,\rangle)$ be relatively compact domains with smooth boundaries. Assume that the scalar curvature $s(x)$ verifies*

$$s(x) \leqslant c^2h(x) \quad on\ M \setminus \Omega_0 \tag{4.33}$$

for some $c > 0$, $h \in C^0(M)$ nonpositive, $h(x) < 0$ on $\overline{\Omega_1} \setminus \Omega_0$. Then there exists $\eta > 0$ such that, if

$$s(x) \leqslant \eta \quad on\ M$$

then there exists a complete conformal metric $\widetilde{\langle\,,\rangle}$ homothetic to $\langle\,,\rangle$ on $M \setminus \Omega_1$, whose scalar curvature $\tilde{s}(x)$ satisfies

$$\tilde{s}(x) \leqslant \tilde{c}^2h(x) \quad on\ M$$

for some $\tilde{c} > 0$. In particular, $\widetilde{\langle\,,\rangle}$ is uniformally equivalent to $\langle\,,\rangle$.

Because of (4.31), $s(x)$ satisfies (4.33) with an appropriate choice of h satisfying the assumptions of Proposition 4.4. Thus we can deform the metric $\langle\,,\rangle$ to a new conformal metric

$$\widetilde{\langle\,,\rangle} = u^{\frac{4}{m-2}}\langle\,,\rangle, \quad u > 0, \tag{4.34}$$

which is homothetic to $\langle\,,\rangle$ on $M \setminus B_R$ and conformally equivalent to it on M. Furthermore its scalar curvature $\tilde{s}(x)$ is non-positive. Let $\tilde{r}(x) = dist_{(M,\widetilde{\langle,\rangle})}(x, o)$. Note that there exists $C \geqslant 1$ such that on M

$$C^{-1}r(x) \leqslant \tilde{r}(x) \leqslant Cr(x). \tag{4.35}$$

Since on $M \setminus B_R$

$$\tilde{s}(x) = \lambda s(x)$$

for some constant $\lambda > 0$, (4.31)(i) and (4.35) imply that

$$\tilde{s}(x) \leqslant \begin{cases} -A\lambda C^{-\alpha}\tilde{r}(x)^{\alpha} & \text{if } \alpha \geqslant 0, \\ -A\lambda C^{\alpha}\tilde{r}(x)^{\alpha} & \text{if } \alpha < 0 \end{cases} \tag{4.36}$$

on $M \setminus B_R$. Similarly, using (4.31)(ii) we obtain

$$K(x) \geqslant \begin{cases} -BC^{\beta}\tilde{r}(x)^{\beta}e^{DC^{\theta}\tilde{r}(x)^{\theta}} & \text{if } \beta \geqslant 0, \\ -BC^{-\beta}\tilde{r}(x)^{\beta}e^{DC^{\theta}\tilde{r}(x)^{\theta}} & \text{if } \beta < 0 \end{cases} \tag{4.37}$$

on $M \setminus B_R$, since $\theta > 0$. Now, with $\lambda > 0$ introduced above, we have

$$\widetilde{Ricci}_{(M,\widetilde{\langle,\rangle})} = T\,Ricci_{(M,\langle,\rangle)} \quad \text{on } M \setminus B_R$$

for some $T > 0$. Thus for any $X \in T_x(M)$

$$\begin{aligned} \widetilde{Ricci}_{(M,\widetilde{\langle,\rangle})}(X, X) &= T\,Ricci_{(M,\langle,\rangle)}(X, X) \\ &\geqslant -T(m-1)H^2\big(1 + r(x)^2\big)^{\frac{\delta}{2}}\langle X, X\rangle \\ &= -\lambda T(m-1)H^2\big(1 + r(x)^2\big)^{\frac{\delta}{2}}\widetilde{\langle X, X\rangle}. \end{aligned}$$

Now because of (4.35), there exists $\hat{T} = \hat{T}(C, \delta) > 0$ such that

$$\widetilde{Ricci}_{(M,\widetilde{\langle,\rangle})} \geqslant -(m-1)\lambda\hat{T}H^2\big(1 + \tilde{r}(x)^2\big)^{\frac{\delta}{2}}\widetilde{\langle X, X\rangle},$$

in other words

$$\widetilde{Ricci}_{(M,\widetilde{\langle,\rangle})} \geqslant -(m-1)\lambda\hat{T}H^2\big(1 + \tilde{r}(x)^2\big)^{\frac{\delta}{2}} \tag{4.38}$$

on $M \setminus B_R$.

We now apply Theorem 4.2 to $(M, \widetilde{\langle\,,\rangle})$ and to the equation

$$c_m \tilde{\Delta} u - \tilde{s}(x)u + K(x)u^{\frac{m+2}{m-2}} = 0, \tag{4.39}$$

noting that (4.9) and (4.10) are satisfied for any $\tilde{\tau}$, since $\tilde{s}(x) \leqslant 0$, while (4.7)(i), (ii) hold because of (4.36) and (4.37) and finally (4.5) is met by (4.38). Therefore we deduce that (4.39) admits a positive solution, in other words there exists a metric $(\,,) = v^{\frac{4}{m-2}} \widetilde{\langle\,,\rangle}$ $(v > 0)$, of scalar curvature $K(x)$, which is a conformal deformation of $\widetilde{\langle\,,\rangle}$. Comparing with (4.34), we have

$$(\,,) = (uv)^{\frac{4}{m-2}} \langle\,,\rangle,$$

which implies that $(\,,)$ is the sought conformal deformation of $\langle\,,\rangle$. $\square$

We observe that the deformed metrics obtained in Theorem 4.3 are not in general complete. For instance assume that

$$Ricci_{(M,\langle\,,\rangle)} \geqslant -(m-1)H^2\big(1 + r(x)^2\big)^{\frac{\delta}{2}} \quad \text{on } M \setminus B_R, \tag{4.40}$$

for some $-2 < \delta < 0$, $H > 0$, and suppose $s(x) \leqslant 0$ on M and

$$s(x) \leqslant -Ar^{\delta}(x) \quad \text{on } M \setminus B_R$$

for some $A \leqslant m(m-1)H^2$. Note that (4.40) yields

$$s(x) \geqslant -A_1 r^{\delta}(x)$$

for some $A \leqslant A_1 \leqslant m(m-1)H^2$. Let $K(x) \in C^{\infty}(M)$ satisfying $K(x) < 0$ on M and

$$-B_1 r^{\beta} e^{Dr^{\theta}} \leqslant K(x) \leqslant -Br^{2+\gamma}(\log r)^{2(1+\epsilon)}$$

on $M \setminus B_R$, with $\gamma \geqslant \delta$. Applying Theorem 4.3 we deduce that the metric $\langle\,,\rangle$ can be conformally deformed to a metric $(\,,) = u^{\frac{4}{m-2}} \langle\,,\rangle$ with scalar curvature $K(x)$. We apply Proposition 4.1 of [13], with the choice $\psi(r(x)) = r(\log r)^{1+\epsilon}$, to deduce that u satisfies

$$u \leqslant C\psi\big(r(x)\big)^{-\frac{m-2}{2}} = C\big[r(\log r)^{(1+\epsilon)}\big]^{-\frac{m-2}{2}} \tag{4.41}$$

for some $C > 0$ and r sufficiently large. This implies that $(\,,)$ is not complete. Indeed let $\{x_n\} \to \infty$ be a sequence in $(M, \langle\,,\rangle)$. Thus $r(x_n) \to +\infty$ and we can suppose that $\{x_n\} \subset M \setminus B_R$ on which (4.41) holds. For n fixed, by Hopf–Rinow theorem, there exists γ_n unit speed geodesic from o to x_n realizing the distance $r(x_n)$. Thus if t is the arclenght parameter of γ_n then

$$r\big(\gamma_n(t)\big) = t.$$

For $\tilde{r}(x) = dist_{(\,,)}(o, x)$ we have

$$\tilde{r}(x_n) \leqslant \int_0^{r(x_n)} u^{\frac{2}{m-2}}\big(\gamma_n(t)\big)\,dt \leqslant C^{\frac{2}{m-2}} \int_0^{r(x_n)} \big[r\big(\gamma_n(t)\big)\big(\log r\big(\gamma_n(t)\big)\big)^{(1+\epsilon)}\big]^{-1}\,dt$$
$$= C^{\frac{2}{m-2}} \int_0^{r(x_n)} \big[t(\log t)^{(1+\epsilon)}\big]^{-1}\,dt \leqslant C^{\frac{2}{m-2}} \int_0^{+\infty} \big[t(\log t)^{(1+\epsilon)}\big]^{-1}\,dt \leqslant \hat{C}$$

with $\hat{C} > 0$ an absolute constant. Thus

$$\tilde{r}(x_n) \leqslant \hat{C} \quad \forall n \in \mathbb{N}$$

and the metric $(\,,\,)$ cannot be complete since $\{x_n\}$ is a divergent sequence in M (i.e. it definitely lies outside any compact set).

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